# *Algebraic Geometry and Topology*

*A SYMPOSIUM IN HONOR OF S. LEFSCHETZ*

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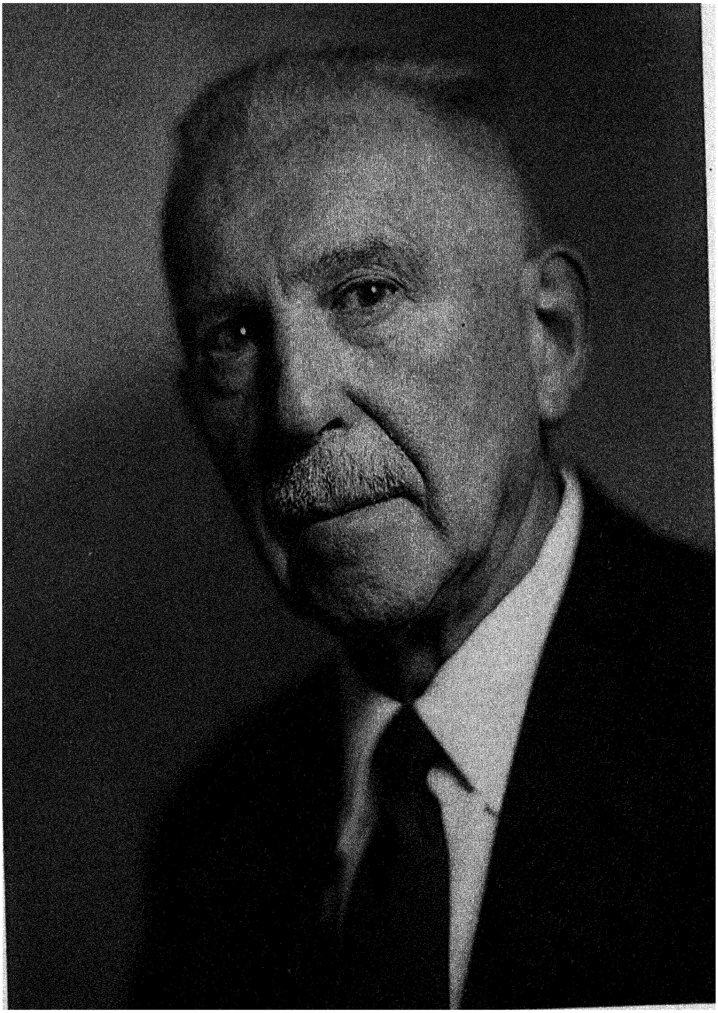

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## *A Symposium in honor of S. Lefschetz*

*Edited by R. H. Fox, D. C. Spencer, A. W. Tucker*
*for the Department of Mathematics*
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# *Foreword*

THIS volume contains papers in Algebraic Geometry and Topology contributed by mathematical colleagues of S. Lefschetz to celebrate his seventieth birthday (September 3, 1954). The goal has been to feature contemporary research that has developed from the vital basic work of Professor Lefschetz.

Eight of the papers, including the special surveys by W. V. D. Hodge and N. E. Steenrod, were presented at the Conference on Algebraic Geometry and Topology held in honor of Professor Lefschetz April 8–10, 1954, at Fine Hall, Princeton University.

The editing of the volume has been a joint enterprise of the members of the Princeton Department of Mathematics. In addition, the following kindly served as referees: E. G. Begle, M. P. Gaffney, Jr., V. K. A. M. Gugenheim, A. P. Mattuck, F. P. Peterson, H. Samelson, E. Snapper, and O. Zariski. The Departmental secretaries, Mrs. Agnes Henry and Mrs. Virginia Nonziato, and the *Annals of Mathematics* secretaries, Mrs. Ellen Weber and Mrs. Bettie Schrader, have ably looked after the many details of correspondence, manuscripts, and proofs. To all these, and to the Princeton University Press and its Director, Mr. H. S. Bailey, Jr., the Editors express their gratitude for the unstinted cooperation that has produced this volume.

A. W. TUCKER

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## *An appreciation of the work and influence of S. Lefschetz*

# *Professor Lefschetz's Contributions to Algebraic Geometry: An Appreciation*

*W. V. D. Hodge*

IF one attempts to give a systematic account of the work of a distinguished mathematician paper by paper, the final impression produced is apt to make this work appear as a museum piece, and this may be the opposite of what is really intended. If any such impression should be given by my remarks on Professor Lefschetz's work on algebraic geometry, it would be particularly unfortunate and misleading; for it is a fact that a number of the discoveries which he has made are of more vital interest to mathematicians at the present day than they have been since they aroused widespread excitement on their first appearance some thirty or thirty-five years ago. I therefore think it would be wiser for me not to follow the historical method, but to select a number of the outstanding contributions which Lefschetz has made to algebraic geometry and to discuss these in relation to current mathematical developments; and I venture to suggest that this procedure will have the approval of Lefschetz himself, since, though no one is more generous in recognizing the merits of others, to him mathematics is more important than the mathematician.

Without a word of warning, however, the method which I propose to follow may equally give a wrong impression, for in treating different aspects of Lefschetz's work separately I may give the idea that the ideas were developed independently, and this is far from being the case. Indeed, one of the most striking features of the whole range of Lefschetz's contributions to the topological and transcendental theories of algebraic varieties is the interplay between the various ideas, and it is clear that the next step in one line was inspired by some achievement in another direction. The essential fact is that Lefschetz clearly regarded the whole range which he covered as a single subject, and the keynote is its unity—a unity which he eventually carried into his work in pure topology, where his fixed-point theorems are clearly derived from ideas familiar in algebraic geometry. But having made this point, it will be convenient to take up a number of aspects of his work separately.

## The topology of algebraic varieties

One of the great lessons to be learned from a study of Lefschetz's work on algebraic varieties is that before proceeding to the investigation of transcendental properties it is necessary first to acquire a thorough understanding of the topological properties of a variety. Not only does this greatly simplify the technical problems encountered in developing the transcendental theory, but it is an absolute necessity if one is to appreciate the true significance of the difficulties to be overcome. It is therefore natural to begin an appreciation of Lefschetz's work with a discussion of his investigations of the topological structure of varieties.

The importance of the topological structure of curves in the study of Abelian integrals of algebraic functions of one variable was made clear by Riemann, and when Picard came to study the integrals attached to an algebraic surface he naturally made use of all the topological methods available. His difficulty, of course, was the fact that at that time our knowledge of topology was extremely primitive, and he had to reinforce his weak topological weapons with more powerful analytic ones. One of the most impressive features of his celebrated treatise is the way in which, by such primitive and indirect means, he did succeed in obtaining a deep understanding of the topological nature of an algebraic surface, though naturally, owing to the use of transcendental methods, some of the finer aspects of the topology, such as torsion, were lost. Nevertheless, his final analysis of the topology of an algebraic surface provided Lefschetz with the scheme for a direct investigation of the topology, not only of surfaces but of varieties of any dimension. Lefschetz's achievement was to obtain all of Picard's topological results by direct methods, and then to use them to simplify the transcendental theory. I wish to emphasize the directness of Lefschetz's methods, for it does seem to me to be most satisfactory to know the topological structure of a variety thoroughly before embarking on transcendental considerations. This is not intended as a criticism of elegant methods which have been employed to obtain results similar to those of Lefschetz by using harmonic integrals or the theory of stacks (though it must be remembered these methods only obtain the homology groups with complex (or real) coefficients, whereas the direct method enables us to use integer coefficients), but merely reflects my view that without a direct method of investigating the topology of an algebraic variety something important would be missing.

The method used by Lefschetz is a direct generalization of the classical process for studying the topology of an ordinary Riemann surface by reducing it to an open 2-cell by means of a system of cuts. Lefschetz similarly introduced cuts into a non-singular variety $V$ of complex dimension $d$ and reduced it to an open $2d$-cell. Representing this $2d$-cell as the interior $C_{2d}$ of a solid $2d$-sphere, the problem is then to determine the image of the boundary of $C_{2d}$ in $V$, which, as a point set, coincides with the cuts. Once this is done, it is possible to read off all the necessary results.

An induction argument is used to reduce $V$ to a cell, making use of the fact that on $V$ there are systems of subvarieties of dimension $d-1$. A suitable subsystem is selected, for example, a pencil $|S_z|$ of prime sections which has a non-singular subvariety $B$ of dimension $d-2$ as base and contains only a finite number of singular sections $S_1, \ldots, S_N$, where the singular point $P_i$ of $S_i$ is not in $B$. The varieties of $|S_z|$ can be represented by the points of the complex plane $\Sigma$, and the singular sections by the critical values $z_1, \ldots, z_N$ of $z$. It is quite simple to prove rigorously that a homeomorphism can be established between two non-singular sections $S_{z'}$ and $S_{z''}$ in which points in $B$ are self-corresponding. What Lefschetz then does is to introduce on $\Sigma$ a set of cuts $z_0 z_i$ $(i = 1, \ldots, N)$ from a suitable point $z_0$ to the critical points $z_i$, and to try to establish a uniquely determined homeomorphism $T_{z'z''}$ between $C_{z'}$ and $C_{z''}$, where $z'$ and $z''$ are in the cut plane $\Sigma'$ with the properties: (a) points in $B$ are unaltered in $T_{z'z''}$, and (b) $T_{z''z'''}T_{z'z''} = T_{z'z'''}$. By the hypothesis of induction, if $z$ is any point in $\Sigma'$, $C_z$ can be reduced to a cell $C'_z$ by introducing cuts in such a way that $B$ lies on the boundary of the cell, and the homeomorphism $T_{zz'}$ enables us to reduce $C_{z'}$ to a cell. Thus we can reduce each $C_z$ $(z \in \Sigma')$ to a cell homeomorphic to $C'_z$, and then $V$ is reduced to a cell.

The next step required is to discuss the nature of the boundary of $C'_z \times \Sigma'$. This requires (a) a discussion of the behavior of $C'_z$ as $z \to z_i$ $(i = 1, \ldots, N)$; the essential point here is that we obtain a mapping of $C_z$ *onto* $C_{z_i}$; (b) a discussion of the homeomorphism induced on $C_{z'}$, where $z'$ is on $z_0 z_i$ and $z$ describes a circuit round $z_i$ beginning and ending at $z'$ going from the right to the left of $z_0 z_i$; in fact, it is sufficient to consider the effect of this homeomorphism on the various homology groups of $C_{z'}$ and this Lefschetz examined in detail. By investigating these questions Lefschetz was able to extract all the information required about the topology of $V$.

The guiding principles of the whole of this investigation are clear; the difficulties lie in the details. It must be recalled that at the time this work was being done, while the study of algebraic topology was getting under way, the topological tools available were still primitive, and one would expect that nowadays, when we have so much more powerful tools at our disposal, it would be possible to use them to straighten out the difficulties which arise over details. It is indeed surprising that we have had to wait almost to the present moment for this to be done. In a forthcoming paper in the ANNALS OF MATHEMATICS, A. H. Wallace has used modern singular homology theory to rewrite that part of Lefschetz's work which deals with the homology groups of dimension less than $d$, and he has shown me a preliminary account of a subsequent paper in which he deals with the remaining homology groups. It is most striking to see how Wallace's methods in principle follow so closely the methods of Lefschetz and indeed provide a triumphant justification of them.

One of the main difficulties of the Lefschetz argument is to establish the unique homeomorphism $T_{z'z''}$ referred to above, for all points $z'$, $z''$ of $\Sigma'$. It is quite easy to do this when $z'$ is any point of $\Sigma'$ different from the $z_i$ and $z''$ is sufficiently near $z'$. This suggests that instead of using direct products we should use fibre bundles. Let $K$ be any closed set in $\Sigma$ not containing a critical point. Wallace shows that it is possible to construct a fibre bundle $X_K$ over $K$, whose fibre is $C_{z'}$, and an embedded bundle $X'_K$ over $K$ equivalent to $B \times K$, such that, if $V_K$ denotes the part of $V$ covered by $C_z$ $(z \in K)$, there is a continuous mapping $\psi$ of $X_K$ onto $V_K$ which, when restricted to $X_K - X'_K$, is a homeomorphism on $V_K - B$, and projects the direct product $X'_K$ on $B$. If $K$ includes a critical point $z_i$, we can construct $X_K$, $X'_K$ as before, but in this case $\psi$ restricted to $X_K - X'_K$ ceases to be a homeomorphism at the part of $X_K - X'_K$ over $z_i$. The behavior of $\psi$ at the critical points can be determined. Then taking $K = \Sigma$ it is possible to determine the singular homology properties of the pair $(X_\Sigma, X'_\Sigma)$ and to use the properties of $\psi$ to deduce the singular homology properties of $V$. With a number of obvious points of difference, Wallace's treatment bears a great resemblance to Lefschetz's original methods.

Thus Lefschetz's pioneer work comes into its own. In addition to its own intrinsic interest, it has a key position in the literature of algebraic varieties as the basis of a great deal of the transcendental theory of varieties. But it also has a unique historical interest, in being almost the first account of the topology of a construct of importance in general mathematics which is not trivial. And it settled a number of questions which now seem trivial, but which at one time caused a good deal of speculation. For instance, the fact the Betti numbers $R_p$ of odd dimension are even, and that $R_p \geqq R_{p-2}$ $(2 \leqq p \leqq m)$, showed at once that not all orientable manifolds of even dimension are the carrier manifolds of algebraic varieties. Moreover, Lefschetz's work is the direct inspiration of all the researches which have taken place subsequently in the theory of complex manifolds. In fact, it is not too much to say that our greatest debt to Lefschetz lies in the fact that he showed us that a study of topology was an essential for all algebraic geometers.

## Integrals of the second kind on an algebraic variety

One of the first applications of his work on the topology of algebraic varieties which Lefschetz made was to the theory of integrals of the second kind. Some of his work on this subject preceded the work on the topology of varieties, and it seems fairly clear that he was led to the topological work in order to make progress possible in the study of integrals. However that may be, it is certain that his most important contribution to our knowledge of integrals of the second kind depends essentially on his previous study of topology. In this he was, essentially, reversing the order followed by Picard, who used the theory of integrals to get to the topology.

It is as well to point out that there are several definitions of integrals of the second kind on an algebraic variety, and that they are not all equivalent. The notion derives, of course, from the notion of an integral of the second kind on a Riemann surface, where we have three possible definitions. It will be recalled that in discussing integrals the fields of integration must be in the open manifold obtained by removing the locus of singularities. The integrals under discussion are assumed to be integrals of forms which are meromorphic everywhere (rational forms).

(a) An integral is of the second kind if all its residues are zero. (A residue is an integral over a cycle which bounds on the Riemann surface);

(b) An integral is of the second kind if in the neighborhood of any point its integrand is equal to the derived of a local meromorphic 0-form (function);

(c) An integral is of the second kind if in the neighborhood of any point its integrand differs from the derived of a rational 0-form on the Riemann surface by a form holomorphic at the point.

The equivalence of these three definitions is almost trivial when $p=1$. But when we extend our investigation to exact $p$-fold integrals on a variety $V$ of dimension $d$, they are no longer equivalent. With trivial alterations in wording, (a) and (b) give unambiguous definitions on $V$; but there are two ways in which (c) can be generalized:

($c_1$) A $p$-fold integral is of the second kind if in the neighborhood of any point its integrand differs from the derived of a rational $(p-1)$-form on $V$ by a locally holomorphic $p$-form;

($c_2$) If $\Gamma_1, \ldots, \Gamma_k$ are the irreducible components of the locus of the singularities of the integral, $\int Q$ is an integral of the second kind if there exists a rational $(p-1)$-form $P_i$ such that $\Gamma_i$ is not a component of the locus of singularities of $Q-dP_i$.

In his investigations Picard used the definition ($c_1$), and Lefschetz uses ($c_2$). In the case $p=2$, $d=2$ both prove that their definitions are equivalent to (a), and hence, indirectly, we deduce the equivalence of ($c_1$) and ($c_2$). But even in this case, (b) is not equivalent to any of the others, and is a much weaker condition. For example, let $f(x,y,z)=0$ be the equation of a surface $V$ of order $n$ with only ordinary singularities and suppose that $x=0$ is a general section of this. Then it is easily seen that if $P(x,y,z)$ is an adjoint polynomial of order $n-3$, the integral

$$\int \frac{P(x,y,z)}{xf_z}\,dx\,dy$$

satisfies condition (b), but has residues, namely, the periods of the integral

$$\int \left(\frac{P(x,y,z)}{f_z}\right)_{x=0} dy$$

on the section $x=0$. More generally, it can be shown that *any* meromorphic integral of multiplicity $q$ ($q \geqq 2$) satisfies condition (b) at any *non-singular* point of the singular locus of the integral. Neither Picard nor Lefschetz ever used (b); at the time at which they worked on this field it was more natural to think always in terms of globally defined forms on a variety, and it is impossible to say whether they were aware of the difficulties which attach to definition (b); but I feel it is necessary to point out the difference between (b) and the others, since a number of modern writers do use the term 'integral of the second kind' for the integral of a form which satisfies (b).

For general values of $p$ and $d$, it has, as far as I am aware, never been proved that ($c_1$) and ($c_2$) are equivalent, though this may well be the case. But Lefschetz has shown that for $p=3$, $d=3$, an integral may have non-zero residues and yet be the integral of a derived form, contrary to what happens when $p=1, 2$. Hence in any discussion of integrals of the second kind it is necessary to make clear at the outset which definition is being used. Each definition can lead to an interesting theory, but one must not expect the same theory from different definitions.

If definition ($c_1$) or ($c_2$) is used, two $p$-fold integrals of the second kind are to be regarded as equivalent if their difference is equal to the derived of a rational $(p-1)$-form on $V$. The main problem is to determine the group of equivalence classes of $p$-fold integrals of the second kind. The first step, both with Picard and with Lefschetz, is to show that any integral of the second kind is equivalent to one having as its singular locus a fixed non-singular prime section $C$ of $V$—usually taken as the section by the prime at infinity. The second stage is to show that any $p$-fold integral of the second kind having $C$ as its complete singular locus which has zero periods on all the $p$-cycles of $V$ in $V-C$ is equivalent to zero, and that there are integrals of the second kind having arbitrarily assigned periods on the $p$-cycles of $V$ in $V-C$.

There is not much to be said here about the case $p=1$, which is really classical. Picard and Lefschetz both treated the case $p=2$, $d=2$ by somewhat similar but not identical methods, and Lefschetz extended the results to the case $p=2$, any $d$. He also gave a brief account of the case $p=3$, $d=3$, along lines which, in theory, should be applicable to any $p$, $d$, though there are a number of difficulties of a topological nature which still require detailed study. This work is extremely ingenious, and merits much more attention than it has ever had. But there is still a third stage in the process of computing the groups of $p$-fold integrals of the second kind which we have still to mention, and it is here that Lefschetz's topological approach produced an urgently needed fresh idea. We have seen that a base for the $p$-fold integrals of the second kind can be constructed by taking a set of $r_p$ integrals having singularities only on $C$ and having independent periods on the $r_p$ independent cycles of $V$ which lie in $V-C$. But these $r_p$ integrals may not be independent. The case $p=2$, $d=2$ was fully investigated by Picard, and the results he obtained constitute one of his most famous contributions to the theory of surfaces. His solution related the non-zero combinations of his $r_p$ integrals whose integrands are derived with the simple integrals of the third kind, but it is more convenient to explain his results in terms of the theory of algebraic equivalence of curves on a surface due to Severi. On the algebraic surface $V$ there exists a set of curves $\Gamma_1, \ldots, \Gamma_\rho$ which are algebraically independent, and are such that any other curve on the surface is algebraically dependent on them. Without loss of generality, we may take $\Gamma_\rho = C$, and suppose that $\Gamma_1, \ldots, \Gamma_{\rho-1}$ are virtual curves of order zero. Then each $\Gamma_i$ $(i \leqq \rho - 1)$ can be represented as a 2-cycle in $V - C$, and a base $\Delta_1, \ldots, \Delta_{r_2}$ for $V - C$ can be chosen so that $\Delta_i = \Gamma_i$ $(i \leqq \rho - 1)$, $(\Delta_i, \Delta_j) = 0$ $(i \leqq \rho - 1, j \geqq \rho)$. Picard's result may be stated as follows: an integral of the second kind with singularities only on $C$ is equivalent to zero if and only if its periods on the cycles $\Delta_i$ $(i \geqq \rho)$ are zero.

The problem of extending this result to $p$-fold integrals on a variety of dimension $d$ is not an easy one, and must have been extremely formidable around 1920, when the topological nature of an algebraic variety was not well understood. The difficulty is that Picard's result, as I have stated it, puts perhaps too much emphasis on the algebraic nature of the exceptional 2-cycles, and this obscures the essential properties of the cycles which make them exceptional. A further complication is due to the accident that the dimension of the period cycles is equal to the dimension of the cycles determined by the singular loci of an integral of the second kind in the case $p = 2$, $d = 2$. Lefschetz succeeded in seeing the essential nature of the exceptional cycles. What he has shown, in fact, is that there exists a maximal subset $\Delta_1, \ldots, \Delta_\sigma$ of $p$-cycles on an algebraic variety $V$ such that, if $\bar{C}$ is any subvariety (possibly reducible) of dimension $d - 1$ on $V$, $\Delta_1, \ldots, \Delta_\sigma$ are homologous on $V$ to cycles of $V - \bar{C}$, and an integral of the second kind is equivalent to zero if and only if it has zero periods on the cycles $\Delta_i$ $(i = 1, \ldots, \sigma)$.

Lefschetz's contribution to the theory of integrals of the second kind may therefore be summed up as follows: (1) by beginning with a clear understanding of the topology of an algebraic surface, he greatly simplified the work of Picard on the double integrals on a surface; (2) in extending the theory to double integrals on a variety of any dimension, and then to triple integrals, he made abundantly clear the pattern which the theory should follow in the case of $p$-fold integrals on a variety of $d$ dimensions. That for many years nothing further was written on the subject was due to the fact that geometers felt they knew all that they wanted to know about integrals of the second kind, though some of the formal proofs were not available, owing to difficulties of a purely technical nature. Just as in the case of the study of the topology of an algebraic variety, I think that most geometers felt that there was little point in pursuing a subject the results of which were clear to them until a new technique appeared which would